June 2010 FP3 [redacted] MA Kyrime2

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1. The line $x = 8$ is a directrix of the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > 0,\ b > 0,$$

and the point (2, 0) is the corresponding focus.

Find the value of a and the value of b.

(5)

Foci $(2,0) \Rightarrow \quad ae = 2$

Directrix $x = 8 \Rightarrow \quad \frac{a}{e} = 8$

$ae \times \frac{a}{e} = 2 \times 8$

$\Rightarrow a^2 = 16 \Rightarrow a = 4$

$e = \frac{1}{2}$

$b^2 = a^2(1 - e^2) = 16(1 - \frac{1}{4})$

$= 16 \times \frac{3}{4} = 12$

$\Rightarrow b = \sqrt{12}$

$\therefore \quad a = 4, b = \sqrt{12}$

2. Use calculus to find the exact value of $\int_{-2}^{1} \frac{1}{x^2+4x+13}\,dx$. **(5)**

$$\int_{-2}^{1} \frac{1}{x^2+4x+13}\,dx = \int_{-2}^{1} \frac{1}{(x+2)^2+9}\,dx$$

$$= \left[\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)\right]_{-2}^{1}$$

$$= \frac{\pi}{12} - 0 = \frac{\pi}{12}$$

3. (a) Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that

$$\cosh 2x = 1 + 2\sinh^2 x$$

(3)

(b) Solve the equation

$$\cosh 2x - 3\sinh x = 15,$$

giving your answers as exact logarithms.

(5)

(a)

$$\text{RHS} = 1 + 2\sinh^2 x = 1 + 2\left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= 1 + \frac{(e^x - e^{-x})^2}{2}$$

$$= 1 + \frac{e^{2x} - 1 - 1 + e^{-2x}}{2}$$

$$= 1 + \frac{e^{2x} + e^{-2x} - 2}{2}$$

$$= 1 + \frac{e^{2x} + e^{-2x}}{2} - \frac{2}{2} = 1 + \frac{e^{2x} - e^{-2x}}{2} - 1$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh(2x) = \text{LHS}$$

as required.

(b)

$$1 + 2\sinh^2 x - 3\sinh x = 15$$

$$\therefore \quad 2\sinh^2 x - 3\sinh x - 14 = 0$$

$$(2\sinh x - 7)(\sinh x + 2) = 0$$

$$\therefore \quad \sinh x = \frac{7}{2}$$

$$\Rightarrow \cancel{\text{arsih}} \quad x = \ln\left(\frac{7+\sqrt{53}}{2}\right)$$

$\cancel{\sinh(x) = -2}$ $\quad \sinh x = -2$

$$\Rightarrow \quad x = \ln(\sqrt{5} - 2)$$

4. $I_n = \int_0^a (a - x)^n \cos x \, dx, \quad a > 0, \quad n \geqslant 0$

(a) Show that, for $n \geqslant 2$,

$$I_n = na^{n-1} - n(n-1)I_{n-2}$$ **(5)**

(b) Hence evaluate $\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^2 \cos x \, dx.$ **(3)**

4. (a). $I_n = \int_0^a (a-x)^n \cos x \, \partial x$

Let $u = (a-x)^n \qquad u' = -n(a-x)^{n-1}$

$v' = \cos x \qquad v = \sin x$

$\therefore I_n = \left[\sin x \,(a-x)^n\right]_0^a + n\int_0^a (a-x)^{n-1} \sin x \, \partial x$

$\therefore \cancel{I_n = 0} \qquad \therefore I_n = n\int_0^a (a-x)^{n-1} \sin x \, \partial x$

Let $u = (a-x)^{n-1} \qquad u' = (1-n)(a-x)^{n-2}$

$v' = \sin x \qquad v = -\cos x$

$\therefore I_n = n\left(\left[-\cos x\,(a-x)^{n-1}\right]_0^a + \int_0^a (1-n)\cos x\,(a-x)^{n-2}\,\partial x\right)$

$\therefore I_n = n\left((0 + a^{n-1}) + (1-n)\, I_{n-2}\right)$

$$\Rightarrow \quad I_n = n a^{n-1} + n(1-n) I_{n-2}$$

$$\therefore \quad I_n = n a^{n-1} - n(n-1) I_{n-2}$$

as required.

(b) $I_2 = 2\left(\frac{\pi}{2}\right) - 2 I_0$

$$= \pi - 2\int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) \cos x \, dx$$

$$\cancel{= \pi - 2\int_0^{\pi/2}}$$

$u = \left(\frac{\pi}{2} - x\right) \quad u' = -1$

$v' = \cos x \quad v = \sin x$

$\cancel{= \pi/2}$

$$= \pi - 2\left(\left[\sin x\left(\frac{\pi}{2} - x\right)\right]_0^{\pi/2} + \int_0^{\pi/2} \sin x \, dx \right)$$

$$= \pi - 2\int_0^{\pi/2} \sin x \, dx$$

$$= \pi - 2\left[-\cos x\right]_0^{\pi/2}$$

$$= \pi - 2$$

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5. Given that $y = (\text{arcosh } 3x)^2$, where $3x > 1$, show that

(a) $(9x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 36y$, **(5)**

(b) $(9x^2 - 1)\frac{d^2y}{dx^2} + 9x\frac{dy}{dx} = 18$. **(4)**

5(a). $y = \left(\text{arcosh}(3x)\right)^2$

$\Rightarrow \frac{\partial y}{\partial x} = \left(2 \text{ arcosh } 3x\right) \times \frac{\partial}{\partial x}(\text{arcosh } 3x)$

consider $\frac{\partial}{\partial x}\left(\text{arcosh} 3x\right)$:

Let $u = \text{arcosh } 3x$

$c^2 - s^2 = 1$

$\therefore s^2 = c^2 - 1$

$\Rightarrow \cosh u = 3x$

$\frac{\partial u}{\partial x} \sinh u = 3$

$\therefore \frac{\partial u}{\partial x} = \frac{3}{\sinh u} = \frac{3}{\sqrt{\cosh^2 u - 1}}$

$\therefore \frac{\partial u}{\partial x} = \frac{3}{\sqrt{9x^2 - 1}}$

$\therefore \frac{\partial}{\partial x}(\text{arcosh } 3x) = \frac{3}{\sqrt{9x^2 - 1}}$

$$\therefore \frac{\partial y}{\partial x} = \left(2\operatorname{arcosh} 3x\right) \times \frac{\partial}{\partial x}\left(\operatorname{arcosh}(3x)\right)$$

$$= 2\operatorname{arcosh} 3x \times \frac{3}{\sqrt{9x^2-1}}$$

$$\therefore \frac{\partial y}{\partial x} = \frac{6\operatorname{arcosh} 3x}{\sqrt{9x^2-1}}$$

$$\Rightarrow \left(\frac{\partial y}{\partial x}\right)^2 = \frac{36\left(\operatorname{arcosh} 3x\right)^2}{9x^2-1}$$

$$\therefore (9x^2-1)\left(\frac{\partial y}{\partial x}\right)^2 = \left(9x^2-1\right)\frac{36\left(\operatorname{arcosh} 3x\right)^2}{9x^2-1}$$

$$= 36\left(\operatorname{arcosh} 3x\right)^2$$

$$= 36y^2$$

as required.

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Question 5 continued

(b) $\frac{\partial y}{\partial x} = \frac{6\operatorname{arcosh}3x}{\sqrt{9x^2-1}}$

~~$\therefore \frac{\partial^2 y}{\partial x^2} = (\sqrt{9x^2-1})\left(\frac{18}{\sqrt{9x^2-1}}\right) - 6\operatorname{arcosh}3x$~~

Via quotient rule...

$\therefore$

$$\frac{\partial^2 y}{\partial x^2} = \frac{(\sqrt{9x^2-1})\left(\frac{18}{\sqrt{9x^2-1}}\right) - 6\operatorname{arcosh}3x\left(\frac{1}{2}(9x^2-1)^{-1/2}\cdot 18x\right)}{9x^2-1}$$

$$\therefore \frac{\partial^2 y}{\partial x^2} = \frac{18 - 54x\operatorname{arcosh}3x\,(9x^2-1)^{-1/2}}{9x^2-1}$$

$$\Rightarrow (9x^2-1)\frac{\partial^2 y}{\partial x^2} = (9x^2-1)\,\frac{18-54x\operatorname{arcosh}3x\,(9x^2-1)^{-1/2}}{9x^2-1}$$

$$(9x^2-1)\frac{\partial^2 y}{\partial x^2} = 18-54x\operatorname{arcosh}3x\cdot(9x^2-1)^{-1/2}$$

$$= 18-\frac{54x\operatorname{arcosh}3x}{\sqrt{9x^2-1}}$$

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Question 5 continued

& $$9x\frac{\partial y}{\partial x} = 9x \cdot \frac{6\operatorname{arcosh}3x}{\sqrt{9x^2-1}}$$

$$\therefore 9x\frac{\partial y}{\partial x} = \frac{54x\operatorname{arcosh}3x}{\sqrt{9x^2-1}}$$

$$\therefore LHS = (9x^2-1)\frac{\partial^2 y}{\partial x^2} + 9x\frac{\partial y}{\partial x}$$

$$= 18 - \frac{54x\operatorname{arcosh}3x}{\sqrt{9x^2-1}} + \frac{54x\operatorname{arcosh}3x}{\sqrt{9x^2-1}}$$

$$= 18 + \frac{0}{\sqrt{9x^2-1}}$$

$$= 18 = RHS$$

as required.

Q5

(Total 9 marks)

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6. $$\mathbf{M} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix}, \text{ where } k \text{ is a constant.}$$

Given that $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$ is an eigenvector of **M**,

(a) find the eigenvalue of **M** corresponding to $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$, **(2)**

(b) show that $k = 3$, **(2)**

(c) show that **M** has exactly two eigenvalues. **(4)**

A transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by **M**.

The transformation T maps the line l_1, with cartesian equations $\frac{x-2}{1} = \frac{y}{-3} = \frac{z+1}{4}$, onto the line l_2.

(d) Taking $k = 3$, find cartesian equations of l_2. **(5)**

6(a). $Mx = \lambda x$

$$\therefore \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 6\lambda \\ \lambda \\ 6\lambda \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 24 \\ 4 \\ 6k+6 \end{pmatrix} = \begin{pmatrix} 6\lambda \\ \lambda \\ 6\lambda \end{pmatrix} \Rightarrow \lambda = 4$$

E. Value $= 4$

(b) $$Mx = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 24 \\ 4 \\ 6k+6 \end{pmatrix} = \begin{pmatrix} 6\lambda \\ \lambda \\ 6\lambda \end{pmatrix} = \begin{pmatrix} 24 \\ 4 \\ 24 \end{pmatrix}$$

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Question 6 continued

$\therefore \quad 6k+6=24 \Rightarrow k=\frac{24-6}{6}$

$\Rightarrow k=3$ as required

(c) $A-\lambda I=\begin{pmatrix}1-\lambda & 0 & 3\\ 0 & -2-\lambda & 1\\ 3 & 0 & 1-\lambda\end{pmatrix}$

$\det(A-\lambda I)=0$

$\Rightarrow \det(A-\lambda I)=(1-\lambda)(-2-\lambda)(1-\lambda)+3(-3(-2-\lambda))$

$=(-2-\lambda)\left[(1-\lambda)^2-9\right]=0$

$= \cancel{(-2-\lambda)(\lambda^2-2\lambda+10)}$ $\Rightarrow -2-\lambda=0$

$= -(2+\lambda)(\lambda+2)(\lambda-4)$ $\Rightarrow \lambda=-2$ is an Eg. Value.

$=-(\lambda+2)^2(\lambda-4)=\cancel{-(\lambda+2)(\lambda-4)}$ $\cancel{(1-\lambda)^2+9=0}$

$\cancel{\Rightarrow (1-\lambda)^2=-9}$

$\Rightarrow \lambda=-2$, $\lambda=4$ are only Eg. Vals. $=\cancel{-(\lambda+2)}$ ~~$\therefore$ no solution since discriminant < 0.~~

$\therefore$ only eigen values are

$\lambda = -2$ and

$\lambda = 4$

(d) $x - 2 = -\frac{1}{3}y$, $-\frac{1}{3}y = \frac{z}{4} + \frac{1}{4}$

$x = 2 - \frac{y}{3}$ ~~& $x-2 = \frac{z}{4} + \frac{1}{4}$~~

~~$\Rightarrow 2 - \frac{y}{3} - 2 = \frac{z}{4} + \frac{1}{4}$~~

$z + 1 = -\frac{4}{3}y$

$\therefore$ ~~$z = y$~~

$\therefore z = -1 - \frac{4}{3}y$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - \frac{y}{3} \\ y \\ -1 - \frac{4}{3}y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} -1/3 \\ 1 \\ -4/3 \end{pmatrix}$$

Let $y = \lambda \Rightarrow \underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1/3 \\ 1 \\ -4/3 \end{pmatrix}$

$\therefore \underset{\sim}{r} = \begin{pmatrix} 2 - \lambda/3 \\ \lambda \\ -1 - 4\lambda/3 \end{pmatrix}$ is eqn of l_1

$x = -1 - \frac{13}{3}\lambda \Rightarrow -\frac{3(x+1)}{13} = \lambda$

$y = -\frac{10}{3}\lambda - 1 \Rightarrow \frac{3(y+1)}{-10} = \lambda$

~~$z = 5\lambda$~~ $z = 5 - \frac{7\lambda}{3} \Rightarrow \frac{3(5-z)}{7} = \lambda$

$\therefore \quad -\frac{3x+3}{13} = \frac{3y+3}{-10} = \frac{15-3z}{7}$

~~3~~

$-\frac{(x+1)}{13} = \frac{y+1}{-10} = \frac{5-z}{7}$

7. The plane Π has vector equation

$$\mathbf{r} = 3\mathbf{i} + \mathbf{k} + \lambda(-4\mathbf{i} + \mathbf{j}) + \mu(6\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

(a) Find an equation of Π in the form $\mathbf{r.n} = p$, where $\mathbf{n}$ is a vector perpendicular to Π and p is a constant.
(5)

The point P has coordinates $(6, 13, 5)$. The line l passes through P and is perpendicular to Π. The line l intersects Π at the point N.

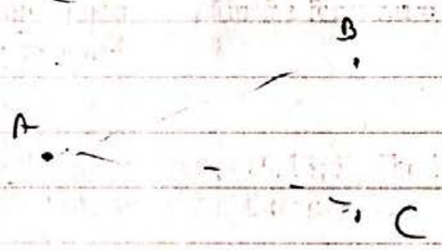
(b) Show that the coordinates of N are $(3, 1, -1)$.
(4)

The point R lies on Π and has coordinates $(1, 0, 2)$.

(c) Find the perpendicular distance from N to the line PR. Give your answer to 3 significant figures.
(5)

Let:

(a) $\underset{\sim}{A} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ $\quad \underset{\sim}{B} - \underset{\sim}{A} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ $\quad \underset{\sim}{C} - \underset{\sim}{A} = \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix}$



$\cancel{\overrightarrow{AB} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}}$ $\quad \overrightarrow{AB} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix}$

$\therefore AB \times AC = \begin{matrix} -4 & 1 & 0 & -4 & 1 & 0 \\ 6 & -2 & 1 & 6 & -2 & 1 \end{matrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ $\quad \Rightarrow \underset{\sim}{n} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

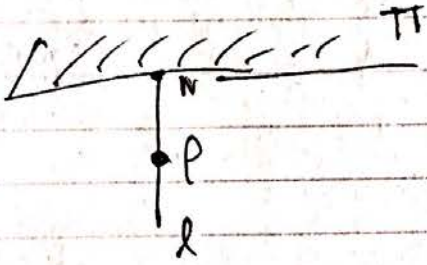
Question 7 continued

$\therefore \underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{A} \cdot \underset{\sim}{n}$

$\therefore \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 5$

(b)



line L has direction vector $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

eqn:

$\therefore \underset{\sim}{r} = \begin{pmatrix} 6 \\ 13 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

N lies on $l \Rightarrow N = \begin{pmatrix} 6+t \\ 13+4t \\ 5+2t \end{pmatrix}$ for some t.

$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 5 \Rightarrow \begin{pmatrix} 6+t \\ 13+4t \\ 5+2t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = 5$

$\therefore 6 + t + 52 + 16t + 10 + 4t = 5$

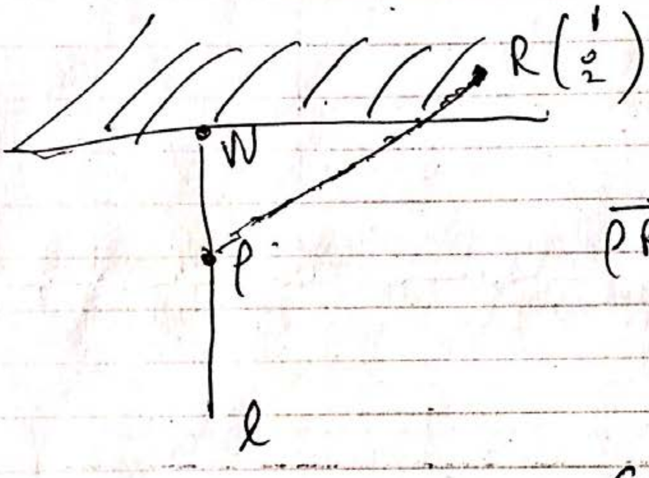
$\Rightarrow 68 + 21t = 5 \quad \Rightarrow t = -3$

Question 7 continued

$$\therefore N = \begin{pmatrix} 6-3 \\ 13+4(-3) \\ 5+2(-3) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

as required.

(c)

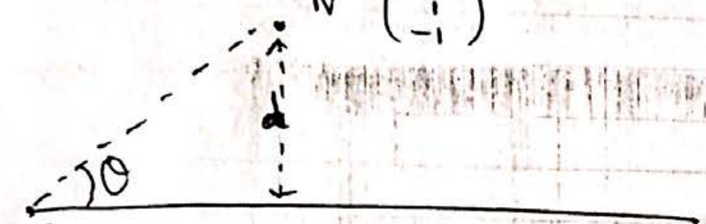


$$\overrightarrow{PR} = \begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}6\\13\\5\end{pmatrix} = \begin{pmatrix}-5\\-13\\-3\end{pmatrix}$$

Line $\overrightarrow{PR}$ has eqn $\underset{\sim}{r} = \begin{pmatrix}6\\13\\5\end{pmatrix} + s\begin{pmatrix}-5\\-13\\-3\end{pmatrix}$

$x = 6 - 5s \Rightarrow \frac{6-x}{5} = s$

$y = 13 - 13s \Rightarrow 13 - y = s$



$N\begin{pmatrix}3\\1\\-1\end{pmatrix}$ $\quad P\begin{pmatrix}6\\13\\5\end{pmatrix}$ $\quad R=\begin{pmatrix}1\\0\\2\end{pmatrix}$ $\quad l=\begin{pmatrix}6\\13\\5\end{pmatrix}+s\begin{pmatrix}-5\\-13\\-3\end{pmatrix}$

$\vec{PN}=\begin{pmatrix}-3\\-12\\-6\end{pmatrix}$ & $\vec{PR}=\begin{pmatrix}-5\\-13\\-3\end{pmatrix}$

$$\therefore \quad \cos\theta = \frac{\begin{pmatrix}-3\\-12\\-6\end{pmatrix}\cdot\begin{pmatrix}-5\\-13\\-3\end{pmatrix}}{\sqrt{3^2+12^2+6^2}\times\sqrt{5^2+13^2+3^2}}$$

$$\cos\theta = \frac{189}{21\sqrt{87}}$$

$$\Rightarrow \quad \sin\theta = \frac{\sqrt{58}}{29} = \frac{d}{|PN|}$$

$$|PN| = 3\sqrt{21} \quad \Rightarrow \quad d = 3\sqrt{21}\sin\theta = 3.61$$

$\therefore$ $d = 3.61$ units.

Q1

(Total 5 marks)

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8. The hyperbola H has equation $\frac{x^2}{16}-\frac{y^2}{4}=1$.

The line l_1 is the tangent to H at the point $P(4\sec t, 2\tan t)$.

(a) Use calculus to show that an equation of l_1 is

$$2y\sin t = x - 4\cos t$$

(5)

The line l_2 passes through the origin and is perpendicular to l_1.

The lines l_1 and l_2 intersect at the point Q.

(b) Show that, as t varies, an equation of the locus of Q is

$$(x^2+y^2)^2 = 16x^2 - 4y^2$$

(8)

8(a). $\frac{x^2}{16} - \frac{y^2}{4} = 1$

$\frac{1}{8}x - \frac{1}{2}y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2x}{8y} = +\frac{x}{4y}$

$\therefore \frac{dy}{dx} = \frac{2x}{8y} = \frac{x}{4y}$

$\therefore \left(\frac{dy}{dx}\right)_P = \frac{4\sec t}{8\tan t} = \frac{\sec t}{2\tan t}$

$= \frac{1/\cos t}{2\frac{\sin}{\cos t}}$

$= \frac{1}{2\sin t}$

$m_T = \frac{1}{2}\operatorname{cosec} t$

$\therefore \quad y - y_1 = m(x - x_1)$

$\therefore \quad y - 2\tan t = \frac{1}{2}\operatorname{cosec} t\,(x - 4\sec t)$

$= \frac{x}{2}\operatorname{cosec} t - 2\operatorname{cosec} t \sec t$

$2y\sin t - \frac{4\sin^2 t}{\cos t} = x - 4\sec t$

$\therefore \quad 2y\sin t = x - 4\sec t + \frac{4\sin^2 t}{\cos t}$

$\therefore \quad 2y\sin t = x + \frac{4\sin^2 t - 4}{\cos t}$

$\therefore \quad 2y\sin t = x + \frac{4(\sin^2 t - 1)}{\cos t}$

$\therefore \quad 2y\sin t = x - \frac{4(1 - \sin^2 t)}{\cos t}$

$\therefore \quad 2y\sin t = x - 4 \cdot \frac{\cos^2 t}{\cos t}$

$\therefore \quad 2y\sin t = x - 4\sec t$

as required.

Question 8 continued

$\therefore \quad 2y\sin t = x - 4\cos t$ as required.

(b) Gradient of $l_1 = \dfrac{-1}{\frac{1}{2}\operatorname{cosec} t} = -2\sin t$

$\therefore l_1$ has eqn: $y = -2\sin t\, x$

& l_2 has eqn: $2y\sin t = x - 4\cos t$

$\Rightarrow \quad -4\sin^2 t\, x = x - 4\cos t$

$\therefore \quad (-4\sin^2 t - 1)x = -4\cos t$

$\therefore \quad x = \dfrac{4\cos t}{1 + 4\sin^2 t}$

$\therefore y = -\dfrac{8\sin t\cos t}{1 + 4\sin^2 t}$

$\therefore Q\left(\dfrac{4\cos t}{1 - 4\sin^2 t}, \dfrac{-8\sin t\cos t}{1 + 4\sin^2 t}\right)$

$x^2 = \dfrac{16\cos^2 t}{(1 + 4\sin^2 t)^2}$ & $y^2 = \dfrac{64\sin^2 t\cos^2 t}{(1 + 4\sin^2 t)^2}$

(Total 13 marks)

TOTAL FOR PAPER: 75 MARKS

$$LHS = x^2 + y^2 = \frac{16\cos^2 t + 64\sin^2 t\cos^2 t}{(1+4\sin^2 t)^2}$$

$$\therefore x^2+y^2 = \frac{16\cos^2 t(1+4\sin^2 t)}{(1+4\sin^2 t)^2}$$

$$\therefore x^2+y^2 = \frac{16\cos^2 t}{1+4\sin^2 t}$$

$$\therefore (x^2+y^2)^2 = \frac{256\cos^4 t}{(1+4\sin^2 t)^2}$$

$$RHS: 16x^2 - 4y^2 = \frac{256\cos^2 t}{(1+4\sin^2 t)^2} - \frac{256\sin^2 t\cos^2 t}{(1+4\sin^2 t)^2}$$

$$= \frac{256\cos^2 t(1-\sin^2 t)}{(1+4\sin^2 t)^2}$$

$$= \frac{256\cos^4 t}{(1+4\sin^2 t)^2} = LHS$$ as required.